Deterministic Variability Analysis for Intermediate Storage in Noncontinuous Processes:

Part II: Storage Sizing for Serial Systems

The allowability conditions derived in Part I of this series are applied to develop intermediate storage sizing expressions for serial systems subjected to process parameter variations. Multiple variations in either starting moments, transfer flow rates, or transfer fractions are considered first. These results are then combined using a worst-case analysis to develop size estimates under general variations. A simple statistical basis is given for the bounds on the cumulative variations which are employed in the sizing expressions. An example serial system is discussed to illustrate the results.

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SCOPE

In Part I we developed allowability conditions for various types of single and multiple variations in process parameters which can occur in *L-M* or 1-1 systems involving intermediate storage. These sufficient conditions were expressed in terms of the periodic holdup functions which result after the variation transients have terminated. Although such allowability conditions constitute important theoretical intermediates, they do not directly address the practical issue of providing estimates of the storage size needed to attain continuity of operations. This issue is addressed in the present work. Specifically, we will employ the allowability conditions together with the deterministic sizing results of Karimi and Reklaitis (1983) to deduce expressions for the intermediate storage capacity and initial inventory required to accommodate any specified range of process parameter variations. The sizing expressions which will be developed are straightforward and conservative, but they eliminate the need to consider a large number of specific postulated combinations of sets of process variations.

The previous literature of sizing intermediate storage to accommodate process variability is confined to the work of Takamatsu et al. (1984). In that paper a system involving L identical upstream batch units, intermediate storage, and M identical downstream units was considered under the assumption that parallel units operate with equally spaced starting moments. Variations in the starting moments of a stage and transfer fraction variations were considered and storage sizing expressions were developed. This paper may in part be viewed as an extension and in part as a complement to the previous work.

CONCLUSIONS AND SIGNIFICANCE

Intermediate storage plays a vital role in enhancing the flexibility of noncontinuous plants. The analysis presented in this paper quantifies the flexibility that storage capacity can offer by providing estimates of the incremental storage required to accommodate specified ranges of process parameter variations. The results are confined to serial systems; however, both sets of homogeneous elementary variations and sets of general variations involving starting times, batch sizes, and transfer flow rates are treated. The results for general variations are conservative in treating the components of the variations separately and in assigning the vessel size so as to accommodate jointly the worst instance of each variation. This does however simplify the resulting calculations considerably.

In addition, a simple statistical basis is given for calculating the variation bounds used in the sizing formulae. Thus, all quantities required in the calculations can either be readily obtained from normal operational data or can unambiguously be selected by the designer. The statistical analysis also shows quite clearly the critical importance of the choice of length of operating period on the sizing calculation.

This work, together with its companion paper, Part I, thus constitutes a comprehensive and detailed treatment of deterministic variability analysis for serial periodic systems.

SIZING UNDER NOMINAL CONDITIONS

Consider the periodic 1-1 system described by the equation,

$$\frac{dV(t)}{dt} = \sum_{i=1}^{2} F_i(t - t_{i0}) \tag{1}$$

Assume that the periods ω_1 and ω_2 of the upstream and downstream units are such that there exist integers β_1 and β_2 satisfying

$$\beta_1\omega_1 = \beta_2\omega_2$$

For such a periodic process, Karimi and Reklaitis (1983) have shown the following:

(i) The holdup function is given by

$$V(t) = V(0) + I(t)$$

where,

$$I(t) = \int_0^t \sum_{i=1}^{2} F_i(\tau - t_{i0})d\tau$$

(ii) The required storage vessel size is given by

$$V^* = V(0) + V_{\max} \tag{2}$$

(iii) The initial inventory must satisfy

$$V(0) \geq -V_{\min} \tag{3}$$

where,

$$V_{\max} = \max_t I(t) \text{ and } V_{\min} = \min_t I(t) \tag{4}$$

Moreover, for $t_{10} = 0$ the definitions of $V_{\max}$ and $V_{\min}$ have been reduced to the following closed-form working expressions:

$$V_{\max} = V_1(1 - x_1) + V_2\left[y_2 - y_2' + \frac{RS_{\max} - s}{x_2}\right] \tag{5a}$$

$$V_{\min} = -V_2(1 - x_2) + V_2\left[y_2 - z_2' + \frac{RS_{\min} - s}{x_2}\right] \tag{5b}$$

the various parameters of which are defined in Table 1. Note again that these sizing formulae apply to the 1-1 system operating periodically with nominal system parameter values.

SIZING UNDER ELEMENTARY VARIATIONS

As discussed in Part I, arbitrary variations in process parameters about their nominal values can be studied in terms of three elementary types of variations. We will analyze each of these in turn and then will proceed to the general case.

TABLE 1. DEFINITIONS OF PARAMETERS FOR 1-1 SYSTEM

$\beta_1\omega_1 = \beta_2\omega_2$	$p = 1/\beta_1$
$m = \omega_1/\omega_2$	$\delta_1 = \text{mod}(mx_1,p)$
$r = U_1/U_2$	$R = \min(1,r)$
$y_2 = \text{mod}(t_{20} - t_{10},\omega_2)/\omega_2$	$y_2' = \text{mod}(y_2 - \delta_1,p)$
$z_2 = \text{mod}(x_2 + y_2,1)$	$z_2' = \text{mod}(z_2,p)$
$s = \max(0,x_2 + y_2 - 1)$	$x = \max(x_1,x_2)$
$S_{\max} = \max[0,y_2' - p(1 - x)]$	$S_{\min} = \min[z_2',px]$

Starting Moment Revisions

Consider a serial system with initial starting moments t_{i0} and initial inventory V_0 at time t_0. Suppose that a total of k starting moment revisions have taken place with k_1 of these revisions associated with the first unit and k_2 with the second. Let $t_{ik_i} = t_{i0} + \sum_{j=1}^{k_i}\Delta t_i^j$ represet the cumulative sum of the starting moment revisions for the ith unit. From proposition II of Part I, the holdup after k revisions must be given by

$$V^k(t) = V_0 + \sum_{i=1}^{2}\int_{t_{ik_i}}^{t} F_i(\tau - t_{ik_i})d\tau$$

We seek to derive an expression for the storage size which can accommodate the holdup function $V^k(t)$. By rearranging $V^k(t)$ and shifting the origin to t_{1k_1}, one can readily show that

$$V^k(t) = V_0 + \int_0^{-(t_{2k_2}-t_{1k_1})} F_2(\tau)d\tau + \int_0^t [F_1(t) + F_2(\tau - t_{2k_2} + t_{1k_1})]d\tau$$

By comparison with Eq. 1, it is evident that $V^k(t)$ represents a system with $t_{10} = 0$, $t_{20} = t_{2k_2} - t_{1k_1}$, and an initial holdup given by

$$V^k(0) = V_0 + \int_0^{-(t_{2k_2}-t_{1k_1})} F_2(\tau)d\tau$$

Therefore, from Eq. 2, the storage volume required for the continuability of $V^k(t)$ is

$$V^* = V_0 + \int_0^{-(t_{2k_2}-t_{1k_1})} F_2(\tau)d\tau + V_{\max} \tag{6a}$$

and from Eq. 3 the initial inventory must satisfy,

$$V_0 \geq -V_{\min} - \int_0^{-(t_{2k_2}-t_{1k_1})} F_2(\tau)d\tau \tag{6b}$$

where $V_{\max}$ and $V_{\min}$ are evaluated by using $t_{20} = t_{2k_2} - t_{1k_1}$ in Eqs. 5a and 5b, respectively.

The remaining term requiring simplification is the integral involving $F_2(\tau)$. As shown in Appendix I, using Fourier series constructions this term reduces to

$$\int_0^{-(t_{2k_2}-t_{1k_1})} F_2(\tau)d\tau = \left[\frac{t_{2k_2} - t_{1k_1}}{\omega_2} - y_2 + \frac{s}{x_2}\right]V_2 \tag{7}$$

where,

$$y_2 = \text{mod}(t_{2k_2} - t_{1k_1},\omega_2)/\omega_2$$

Substituting this value of the integral into Eqs. 6a and 6b and combining with the $V_{\max}$ and $V_{\min}$ expressions, we obtain the results summarized in proposition I:

Proposition I. The storage volume, V^*, and initial inventory, V_0, required for the continuability of periodic operation of the 1-1 system after k starting moment revisions are:

$$V^* = V_0 + V_1(1 - x_1) + V_2 f(t_{2k_2} - t_{1k_1}) \tag{8a}$$

and

$$V_0 \geq V_2(1 - x_2) - V_2 g(t_{2k_2} - t_{1k_1}) \qquad (8b)$$

where

$$f(t_{2k_2} - t_{1k_1}) = \frac{t_{2k_2} - t_{1k_1}}{\omega_2} + \frac{RS_{max}}{x_2} - y_2' \qquad (9a)$$

$$g(t_{2k_2} - t_{1k_1}) = \frac{t_{2k_2} - t_{1k_1}}{\omega_2} + \frac{RS_{min}}{x_2} - z_2' \qquad (9b)$$

It is important to note that proposition I only provides the storage volume needed for continuability of $V^k(t)$ after the kth revision. Yet, the allowability theorem (theorem I of Part I) requires continuability of $V^k(t)$ after each successive revision in the set of k revisions. Thus, in principle, V^* must be recalculated using Eqs. 8a and 8b after each successive revision and then the storage size selected to be the largest of all of them. Moreover, even for a fixed k this procedure would have to be repeated for various revision sequences involving different values of k_1 and k_2. Clearly, such an exhaustive approach is quite impractical.

A reasonable estimate of the storage size required for a general set of starting moment revisions can be obtained by further study of the properties of Eqs. 8a and 8b. Observe first of all that the storage vessel size required to accommodate $V^k(t)$ depends upon the cumulative sum of the starting moment revisions. Secondly, the cumulative revision for an individual unit is not itself size-determining; rather, the difference between the sums of revisions for the up and downstream unit is the size-determining factor. Furthermore, as indicated by the following lemma, the size is monotonic in this difference.

Lemma I. The functions f and g of Eqs. 9a and 9b are nondecreasing functions of $t_{2k_2} - t_{1k_1}$.

(For the proof see Appendix II.)

This monotonicity property suggests that if, as basis for a design, one were to specify upper and lower bounds Δt_U and Δt_L on the difference $t_{2k_2} - t_{1k_1}$, that is,

$$\Delta t_L \leq t_{2k_2} - t_{1k_1} \leq \Delta t_U \qquad (10)$$

then one could calculate a worst-case V^* which would accommodate any general set of starting moment revisions whose successive revisions would fall within these bounds. In fact, by using lemma I and these bounds together with proposition I, the following result is readily proven.

Proposition II. All sets of starting moment revisions, such that inequalities (Eq. (10) are satisfied after each successive revision in the set, will be allowable with a storage size given by

$$V^* = V_0 + V_1(1 - x_1) + V_2 f(\Delta t_U) \qquad (11a)$$

and initial inventory satisfying

$$V^0 \geq V_2(1 - x_2) - V_2 g(\Delta t_L) \text{ and } V^0 \geq 0 \qquad (11b)$$

The remaining difficulty is how to assign Δt_U and Δt_L in practice or, more specifically, how to relate these bounds to the likely distribution of variations of the individual unit starting times. The resolution of this issue will be deferred to a later section of the paper. For continuity in the development we instead complete our study of the other types of elementary variations.

Flow Rate Variations

Consider the 1-1 system under a set of variations involving only the flow rates U_1 and U_2. Again suppose that a total of k flow rate variations have taken place with k_i variations associated with the ith unit. From proposition II of Part I, the holdup after k variations is given by

$$V^k(t) = V^0(t) + \sum_{i=1}^{2} c_i \omega_i x_i \sum_{j=1}^{k_i} \Delta U_i^j \qquad (12)$$

In the present instance it is convenient to express the size estimate as an increment ΔV^* over the V^* required in the absence of these variations and, similarly, to express the initial inventory estimate as an increment ΔV_0 over the V_0 required if these variations were absent. In this notation, it follows immediately from Eq. 12 that the increments required to accommodate the $V^k(t)$ holdup function are

$$\Delta V^* = \Delta V_0 + \sum_{i=1}^{2} c_i \omega_i x_i \sum_{j=1}^{k_i} \Delta U_i^j \qquad (13a)$$

and

$$\Delta V_0 \geq - \sum_{i=1}^{2} c_i \omega_i x_i \sum_{j=1}^{k_i} \Delta U_i^j \qquad (13b)$$

For the allowability of a general set of flow rate variations we adopt again the strategy of assuming the possibility of assigning bounds ΔV_L and ΔV_U satisfying

$$\Delta V_L \leq \sum_{i=1}^{2} c_i \omega_i x_i \sum_{j=1}^{k_i} \Delta U_i^j \leq \Delta V_u \text{ for all } k_i$$

where it is assumed that $\Delta V_L \leq 0$ and $\Delta V_U \geq 0$. Given such bounds, Eqs. 13a and 13b can be simply reduced to

$$\Delta V^* = \Delta V_0 + \Delta V_U \text{ and } \Delta V_0 = -\Delta V_L$$

The ΔV_L and ΔV_U bounds can further be expressed in terms of bounds on the cumulative flow rate variations of the individual units. Thus, if bounds $(\Delta U_i)_L$ and $(\Delta U_i)_U$ satisfying

$$(\Delta U_i)_L \leq \sum_{j=1}^{k_i} \Delta U_i^j \leq (\Delta U_i)_U \text{ for all } k_i$$

can be imposed, then it follows that

$$\Delta V_U = \omega_1 x_1 (\Delta U_1)_U - \omega_2 x_2 (\Delta U_2)_L \qquad (14a)$$

and

$$\Delta V_L = \omega_1 x_1 (\Delta U_1)_L - \omega_2 x_2 (\Delta U_2)_U \qquad (14b)$$

The discussion of the relationship between the (ΔU_i) bounds and the likely distribution of variations of the individual unit flow rates will again be deferred to a later section.

Transfer Fraction Variation:

Next, we consider the 1-1 system under a set of variations involving only transfer fractions. The treatment is quite similar to that for flow rate variations. From proposition II of Part I, the holdup after k variations is given by

$$V^k(t) = V^0(t) + \sum_{i=1}^{2} c_i \omega_i U_i \sum_{j=1}^{k_i} \Delta x_i^j$$

and the increments in V^* and V_0 required to accommodate the $V^k(t)$ holdup function are

$$\Delta V^* = \Delta V_0 + \sum_{i=1}^{2} c_i \omega_i U_i \sum_{j=1}^{k_i} \Delta x_i^j$$

and

$$\Delta V_0 \geq - \sum_{i=1}^{2} c_i \omega_i U_i \sum_{j=1}^{k_i} \Delta x_i^j$$

Again assuming the existence of bounds ΔV_L and ΔV_U such that

$$\Delta V_L \leq \sum_{i=1}^{2} c_i \omega_i U_i \sum_{j=1}^{k_i} \Delta x_i^j \leq \Delta V_U$$

with $\Delta V_L \leq 0$ and $\Delta V_U \geq 0$, we have, $\Delta V^* = \Delta V_0 + \Delta V_U$ and $\Delta V_0 = -\Delta V_L$.

These bounds can, as in the case of flow rate variations, be expressed in terms of bounds on the cumulative transfer fraction variations of the individual units. If bounds $(\Delta x_i)_L$ and $(\Delta x_i)_U$ such that

$$(\Delta x_i)_L \leq \sum_{j=1}^{k_i} \Delta x_i^j \leq (\Delta x_i)_U$$

can be assigned, then analogous to Eqs. 14a and 14b we obtain

$$\Delta V_U = \omega_1 U_1 (\Delta x_1)_U - \omega_2 U_2 (\Delta x_2)_L \tag{15a}$$

and

$$\Delta V_L = \omega_1 U_1 (\Delta x_1)_L - \omega_2 U_2 (\Delta x_2)_U \tag{15b}$$

As will be shown later, the bounds $(\Delta x_i)_L$ and $(\Delta x_i)_U$ can be related to the distribution of the variations in the individual unit transfer fractions.

SIZING UNDER GENERAL VARIATIONS

In this section, the previous results obtained for individual elementary variations will be combined to obtain a storage size estimate applicable to general sets of multiple variations. As shown in Part I, composite variations can be decomposed into elementary variations and the elementary variations themselves are superimposable. Using these properties a general holdup function was develop (proposition II) whose continuability insures the allowability of a set of general variations. This holdup function, $V^k(t)$, given below in a rearranged form, constitutes the basis of the present analysis

$$V^k(t) = V_0 + \Delta V + \sum_{i=1}^{2} \int_{t_{ik_{i1}}}^{t} F_i(\tau - t_{ik_{i1}}) d\tau$$

where

$$\Delta V = \sum_{i=1}^{2} c_i \left[\sum_{j=1}^{k_{i3}} \Delta V_i^j + \omega_i (k_{i2} - k_{i3}) \Delta U_i^{k_{i2}} \min(x_i, x_i + \Delta x_i^{k_{i2}}) \right]$$

and

$$\Delta V_i^j = (U_i + \Delta U_i^j)(x_i + \Delta x_i^j)\omega_i - V_i$$

We begin with two simplifying assumptions which reduce the complexity of the analysis of general variations and which effectively make the resulting size estimates more conservative. First, we assume that each of the contributions to the above holdup function are independent of each other. Second, we assume the existence of bounds on the general variations similar to those used in the case of elementary variations. For convenience in analysis we divide the contributions to the function $V^k(t)$ into two parts: the first part is that associated with the starting moment revisions (the integral term), and the second part combines the contributions from batch size variations and flow rate variations (the ΔV term). Corresponding to each of these parts we postulate composite bounds Δt_L, Δt_U, ΔV_L, and ΔV_U satisfying the inequalities,

$$\Delta t_L \leq t_{2k_{21}} - t_{1k_{11}} \leq \Delta t_U \tag{16a}$$

and

$$\Delta V_L \leq \Delta V \leq \Delta V_U \tag{16b}$$

Then we obtain, via a worst-case analysis, the following proposition about the size estimate in the presence of general variations.

Proposition III. The storage size and initial inventory required for the allowability of a set of general variations satisfying the bounds of Eqs. 16a and 16b are given by

$$V^* = V_0 + V_1(1 - x_1) + V_2 f(\Delta t_U) + \Delta V_U \tag{17a}$$

and

$$V_0 = V_2(1 - x_2) - V_2 g(\Delta t_L) - \Delta V_L \tag{17b}$$

Proof. The result is obtained by straightforward superimposition of the largest contributions to the holdup from both parts. Suppose that the combination of starting moment revisions is such that the contribution to the holdup is at its maximum level. Since, from lemma I, f is a nondecreasing function, the maximum contribution occurs when $t_{2k_{21}} - t_{1k_{11}} = \Delta t_U$. Thus, the starting moment variation contribution to the required size will be as in Eq. 11a. Now, superimpose upon this the maximum positive holdup contribution arising from the second part, namely the batch size variations and flow rate variations. The maximum contribution will occur when $\Delta V = (\Delta V)_U$. Finally, to generate the maximum contribution to V^* from the initial inventory, V_0 should be as large as possible. For the part corresponding to the starting time variations this is achieved when $t_{2k_{21}} - t_{2k_{11}} = \Delta t_L$, as in Eq. 11b. For the second part, this is achieved when $\Delta V = \Delta V_L$. Superimposing these contributions, Eq. 17b is obtained.

As in the case of elementary variations, the size estimate V^* provided by proposition III is inseparably linked to the magnitudes of the composite bounds Δt_L, Δt_U, ΔV_L, ΔV_U. The least conservative estimate of V^* will be obtained using estimates of the composite bounds obtained from the joint distributions of the quantities $(t_{2k_{21}} - t_{1k_{11}})$ and ΔV derived from the individual starting time, batch size, and flow rate variations. On the other hand, the most conservative bound will be obtained when the composite bounds are generated by decomposing them into their component variations and by combining the bounds for the variations involving each unit separately. Thus, given cumulative starting moment bounds $(\Delta t_i)_L$ and $(\Delta t_i)_U$ for the individual units which satisfy,

$$(\Delta t_i)_L \leq \sum_{j=1}^{k_{i1}} \Delta t_i^j \leq (\Delta t_i)_U$$

One can obtain the composite bounds

$$\Delta t_L = (\Delta t_2)_L - (\Delta t_1)_U$$
$$\Delta t_U = (\Delta t_2)_U - (\Delta t_1)_L$$

Similarly, given cumulative batch size variation bounds for the individual units $(\Delta V_i)_L$ and $(\Delta V_i)_U$ and individual flow rate variation bounds $(\Delta U_i^j)_L$ and $(\Delta U_i^j)_U$ which satisfy,

$$(\Delta V_i)_L \leq \sum_{j=1}^{k_{i3}} \Delta V_i^j \leq (\Delta V_i)_U$$
$$(\Delta U_i^j)_L \leq \Delta U_i^j \leq (\Delta U_i^j)_U$$

one can obtain the composite bounds

$$\Delta V_L = (\Delta V_1)_L - (\Delta V_2)_U + \omega_i x_i (\Delta U_1^j)_L - \omega_2 x_2 (\Delta U_2^j)_U$$
$$\Delta V_U = (\Delta V_1)_U - (\Delta V_2)_L + \omega_1 x_1 (\Delta V_1^j)_U - \omega_2 x_2 (\Delta U_2^j)_L$$

Regardless of the method used to calculate the composite bounds, these bounds will have to be obtained for specified periods over which uninterrupted operation of the system is desired or for a specified maximum number of difficult possible variations which the design is to accommodate. Clearly, as the length of the time period for which uninterrupted operation is desired is extended, the bounds will increase and hence the storage size required to accommodate variations will increase. This point is further elucidated in the next section, in which we sketch out the relationship between the bounds and the parameters of the distribution functions of individual variations.

STATISTICAL BASIS FOR BOUNDS

In both the case of elementary variations and general variations, it proved convenient to express the sizing results in terms of postulated bounds on cumulative variations. These bounds can be directly related to the statistical parameters of the individual variations under the assumption that the variations are normally distributed provided that the length of the period of uninterrupted operation is specified.

Consider first the case of starting moment revision. Assume that the tank is to be sized to accommodate T time units of uninterrupted operation. This corresponds to $N_i = \text{trunc}(T, \omega_i)$ cycles of operation of the ith unit on an average. In this case, the maximum number of start time revisions possible for the ith unit will be $k_{i1} = N_i - 1$. Now, assume that the starting time revisions Δt_i^j are independent, normally distributed random variables (RV's) with mean zero and standard deviation $\sigma(t_i)$. Clearly the variable χ_1 given by

$$\chi_1 = t_{2k_{21}} - t_{1k_{11}} - t_{20} + t_{10} = \sum_{j=1}^{k_{21}} t_2^j - \sum_{j=1}^{k_{11}} \Delta t_1^j$$

is itself an RV.

Note that since the normal distribution is a symmetric distribution, both Δt_i^j and $-\Delta t_i^j$ have the same distributions. Thus, χ_1 is a sum of independent, normally distributed RV's, and hence it is also a normally distributed RV, as evident from the following lemma (Woodroofe, 1975).

Lemma II. Let $X_1, X_2, \ldots X_n$ be n independent normal RV's. If $S_n = \sum_{i=1}^{n} X_i$ then

$$Pr\left\{\left(S_n - \sum_{i=1}^{n} EX_i\right) \Big/ \left[\sum_{i=1}^{n} \sigma^2(X_i)\right]^{1/2} \leq z\right\} = \Phi(z)$$

where, $\Phi(\cdot)$ is the standard normal distribution function.

Moreover, from elementary statistics, if $X_1, X_2, \ldots, X_n$ are independent normal RV's then mean and variance of RV $Y = \sum_{i=1}^{n} \zeta_i X_i$, are given by,

$$EY = \sum_{i=1}^{n} \zeta_i EX_i$$

$$\sigma^2(Y) = \sum_{i=1}^{n} \zeta_i^2 \sigma^2(X_i)$$

Hence, given the values of k_{i1}, mean and variance of χ_1 are obtained as sums of means and variances of the Δt_i^j. Thus, the mean of χ_1 is zero and its variance is given by

$$\sigma^2(\chi_1) = (k_{21} - 1)\sigma^2(t_2) + (k_{11} - 1)\sigma^2(t_1)$$

In order to estimate the bounds on the possible values of χ_1 with a certain confidence ρ, we employ the following lemma (Petrov, 1975).

Lemma III. Let $X_1, X_2, \ldots X_n$ be n independent normal RV's with $EX_i = 0$. If $S_k = \sum_{i=1}^{k} X_i$, $k = 1, n$ then

$$Pr\left\{\max_{1 \leq k \leq n} |S_k| \geq \tau\right\} \leq 2Pr\{|S_n| \geq \tau\}$$

With the help of the above lemma, one can obtain the bounds on the RV χ_1 in terms of the RV X_1 given by,

$$X_1 = t_{2(N_1-1)} - t_{1(N_1-1)} - t_{20} + t_{10} = \sum_{j=1}^{N_2-1} \Delta t_2^j - \sum_{j=1}^{N_1-1} \Delta t_1^j$$

with a variance,

$$\sigma^2(X_1) = (N_2 - 1)\sigma^2(t_2) + (N_1 - 1)\sigma^2(t_1) \qquad (18)$$

Now, for any desired probability ρ on the bounds of χ_1, one can obtain from standard normal distribution tables the multiplier α such that

$$Pr\{-\alpha\sigma(X_1) \leq X_1 \leq \alpha\sigma(X_1)\} = (1 + \rho)/2 \qquad (19)$$

which will assure that

$$Pr\{-\alpha\sigma(X_1) \leq \chi_1 \leq \alpha\sigma(X_1)\} \geq \rho$$

Setting $(\Delta t)_L = -\alpha\sigma(X_1) + t_{20} - t_{10}$ and $(\Delta t)_U = \alpha\sigma(X_1) + t_{20} - t_{10}$ we have the bounds required by proposition III.

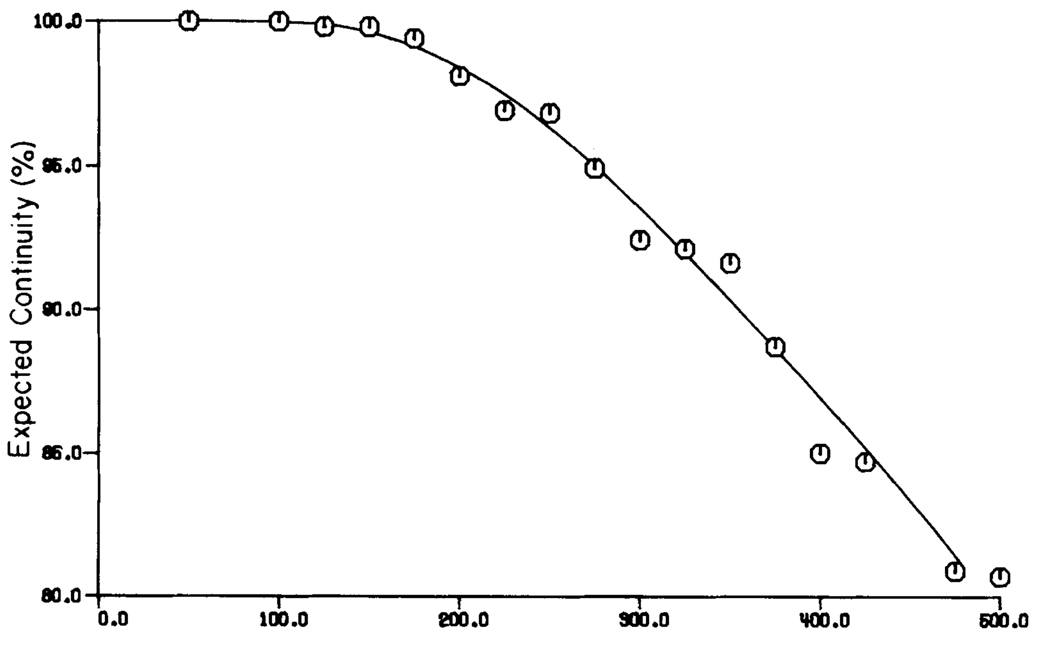

Figure 1. Effect of length of operation on continuity.

If $\sigma(t_i)$ are not directly available, they can be obtained by estimating the likely range of variation of Δt_i^j (e.g., $|\Delta t_i^j| \leq \gamma\omega_i$ where γ is some expected fraction of the cycle time) and estimating the likelihood ρ of starting moment revisions falling within that range. Then $\sigma(t_i)$ can be obtained from the tables by selecting $\sigma(t_i)$ such that

$$Pr\{-\gamma\omega_i \leq \Delta t_i^j \leq \gamma\omega_i\} = \rho$$

The same type of analysis as outlined above for starting moment revisions applies to flow rate variations and transfer fraction variations. In the former case, the random variable in question is

$$X_2 = \omega_2 x_2 \sum_{j=1}^{N_2} \Delta U_2^j - \omega_1 x_1 \sum_{j=1}^{N_1} \Delta U_1^j$$

while in the latter case, it is given by

$$X_3 = \omega_2 U_2 \sum_{j=1}^{N_2} \Delta x_2^j - \omega_1 U_1 \sum_{j=1}^{N_1} \Delta x_1^j$$

Finally, the same calculation holds true for general variations, assuming that the starting moment revisions Δt_i^j, the batch size variations ΔV_i^j, and the flow rate variations ΔU_i^j are all independent, normal RV's.

Two further observations are noteworthy. First, as T increases, the maximum number of variations will increase; hence, from Eq. 18, the variance of the cumulative variation will increase. Consequently, for any fixed probability of successful operation ρ Eq. 19 shows that $(\Delta t)_L$ and $(\Delta t)_U$ will increase and, hence, by virtue of lemma I and proposition III, V^* must be increased. Similarly, for fixed V_0 and V^*, ρ must decrease with increase in T as evident from Figure 1. Clearly the choice of T (or maximum number of expected variations) is an important design parameter. Secondly, although the assumption of normally distributed variations is convenient for this analysis, it is not crucial because as the number of variations become large the variables X_i will approach normal variate properties regardless of the specific distributions of the variations.

EXAMPLE

A serial system has an upstream batch unit and a downstream semicontinuous unit, separated by an intermediate storage facility. The nominal operating parameters of the system are as follows:

Upstream Batch Unit: $V_1 = 8{,}000$ kg, $(T_f)_1 = 2.5$ h, $(T_B)_1 = 7.5$ h, $(T_e)_1 = 2.5$ h, $(T_p)_1 = 0.0$ h, $\omega_1 = 12.5$ h, and $t_{10} = 0.0$ h.
Downstream Semicontinuous Unit: $V_2 = 3{,}200$ kg, $(T_S)_2 = 3.75$ h, $(T_i)_2 = 1.25$ h, $\omega_2 = 5$ h, and $t_{20} = 0.625$ h.

Suppose that with 99% probability the system batch sizes and flow rates will stay within 5% of their nominal values and the system start time revisions will stay within 5% of the system cycle times. Suppose further the system is designed assuming that for 125 hours of operation both the cumulative starting time revisions and the combined batch size and flow rate variations will be accommodated with 90% probability.

From the above data, $x_1 = 0.2$, $x_2 = 0.75$, $U_1 = 3{,}200$ kg/h, and $U_2 = 853.3$ kg/h. Also from Table 1, $m = 2.5$, $x = 0.75$, $r = 3.75$, $R = 1$, $\beta_1 = 2$, $\beta_2 = 5$, $p = 0.5$, $mx_1 = 0.5$ and $\delta_1 = 0$.

We next estimate the variances of the individual variations assuming that they are normally distributed with mean zero. As a consequence of the choice,

$$Pr\{|\Delta t_i^j| \leq 0.05\omega_i\} = 0.99$$

$$Pr\{|\Delta V_i^j| \leq 0.05V_i\} = 0.99$$

and

$$Pr\{|\Delta U_i^j| \leq 0.05U_i\} = 0.99$$

it follows that $\sigma(t_1) = 0.25$ h, $\sigma(t_2) = 0.1$ h, $\sigma(V_1) = 155.3$ kg, $\sigma(V_2) = 62.1$ kg, $\sigma(U_1) = 62.1$ kg/h, and $\sigma(U_2) = 16.6$ kg/h.

For 125 h of operation, the average number of cycles of the first

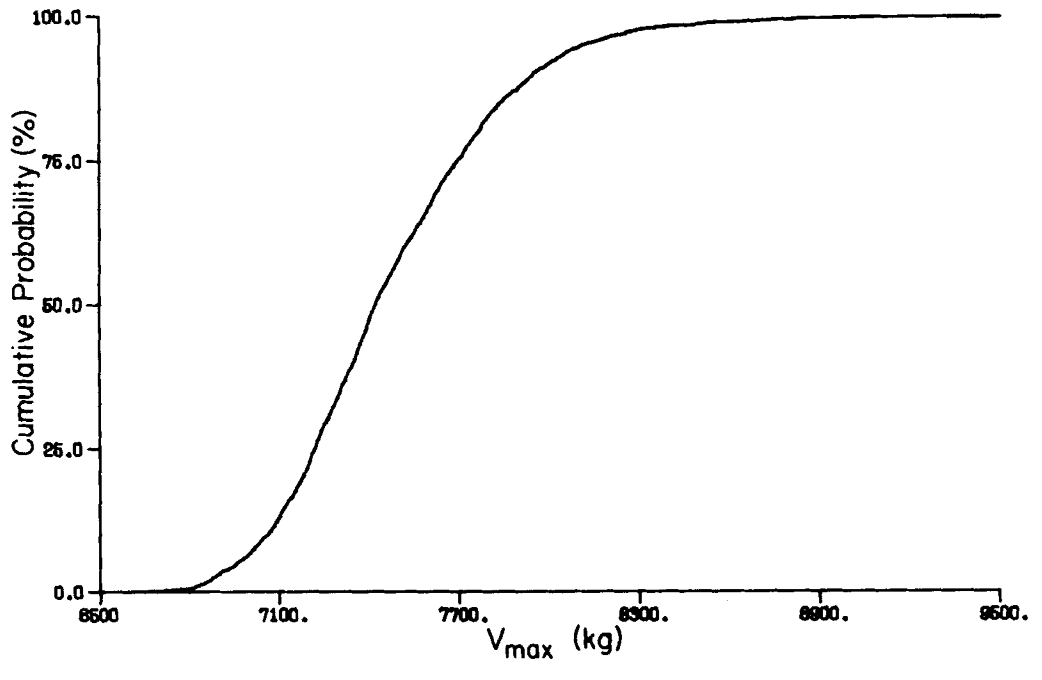

Figure 2. Distribution function for V_{max} computed from simulation.

unit is 10 and of the second unit is 25. Thus $k_{11} = 9$, $k_{21} = 24$, $k_{13} = 10$, and $k_{23} = 25$. The additional contribution due to flow rate variations is at most equal to two variations.

Our aim is to calculate V_0 and V^* such that the continuity of operation is maintained with at least probability $\rho = 0.9$. In other words, we want to calculate the bounds on $\Delta t = t_{2k_{21}} - t_{1k_{11}}$, and ΔV such that

$$Pr\{\Delta t_L \leq \Delta t \leq \Delta t_U, \Delta V_L \leq \Delta V \leq \Delta V_U\} = \rho$$

Since, the RV's Δt and ΔV are assumed to be independent,

$$Pr\{\Delta t_L \leq \Delta t \leq \Delta t_U\} Pr\{\Delta V_L \leq \Delta V \leq \Delta V_U\} = \rho$$

By assuming a balanced design for both types of variations, we have,

$$Pr\{\Delta t_L \leq \Delta t \leq \Delta t_U\} = Pr\{\Delta V_L \leq \Delta V \leq \Delta V_U\} = \sqrt{\rho}$$

Now, for starting moment revisions, the random variable X_1 is given by

$$X_1 = \sum_{j=1}^{24} \Delta t_2^j - \sum_{j=1}^{9} \Delta t_1^j,$$

and its variance is $9\sigma^2(t_1) + 24\sigma^2(t_2) = 0.8\ \text{h}^2$. With the choice of confidence level of $\rho = 0.9$ we obtain from standard normal distribution tables and Eq. 19 that

$$Pr\{-1.75 \leq \chi_1 \leq 1.75\} = 0.949$$

Since $t_{20} - t_{10} = 0.625$, it follows that $\Delta t_L = -1.125$ h and $\Delta t_U = 2.375$ h.

The calculation for the random variable

$$\left(\sum_{j=1}^{10} \Delta V_1^j - \sum_{j=1}^{25} \Delta V_2^j + x_1\omega_1 \Delta U_1^j - x_2\omega_2 \Delta U_2^j\right)$$

is similar and results in the bounds, $\Delta V_L = -1{,}179$ kg and $\Delta V_U = 1{,}179$ kg.

Now, from Table 1, using $\Delta t_U = 2.375$ h for $t_{20} - t_{10}$, we have $y_2 = 0.475$, $y_2' = 0.475$ and $S_{max} = 0.35$. Substituting into Eq. 9a, we obtain $f(\Delta t_U) = 0.47$. Similarly, from $\Delta t_L = -1.125$ h, $y_2 = 0.775$, $z_2 = 0.525$, $z_2' = 0.025$, and $S_{min} = 0.025$. Therefore, from Eq. 9b, $g(\Delta t_L) = -0.22$. Thus, if only starting moment revisions were to occur, $V_0 = 1{,}494$ kg and $V^* = 9{,}388$ kg would be adequate to contain them. However, if all three types of variations must be accommodated then Eq. 17a,b will yield $V_0 = 2{,}673$ kg and $V^* = 11{,}746$ kg. Note that if the storage were designed for nominal values of the system parameters, $V^* = 6{,}400$ kg and no initial inventory would be required.

In order to gauge the conservativeness of our results, we studied the system in this example using a Monte Carlo simulation. From 2,000 simulation runs of 125 h duration each, an estimate of 99.8% was obtained for the probability of uninterrupted operation. This clearly shows that even though the storage vessel was designed to guarantee at least 90% continuity of operation, it is capable of maintaining 99.8% continuity. In order to estimate the required size of storage for 90% continuity, the marginal distribution function $[F(\tau) = Pr\{X \leq \tau\}]$ for initial inventory V_0 (Figure 2) was generated from the 2,000 simulation runs. From Figure 2,

$$Pr\{V_0 \leq 1{,}558\} = 0.95$$

Using this information, the conditional distribution function $[F(\tau) = Pr\{V_{max} \leq \tau \mid V_0 \leq 1{,}558\}$ (Figure 3) was also generated from the simulation. From Figure 3,

$$Pr\{V_{max} \leq 8{,}107 \text{ kg} \mid V_0 \leq 1{,}558 \text{ kg}\} = 0.95$$

therefore, $Pr\{V_0 \leq 1{,}558 \text{ kg}, V_{max} \leq 8{,}107 \text{ kg}\} = 0.90$, and $V_0 = 1{,}558$ kg and $V^* = 9{,}665$ kg are sufficient to insure continuous operation at least 90% of the time. This amounts to an overdesign of 22% in the results from our analysis.

Figure 1 shows the effect of the operation time limit T on the expected continuity of operation, in which the solid line is just a smooth curve through points representing the values obtained from

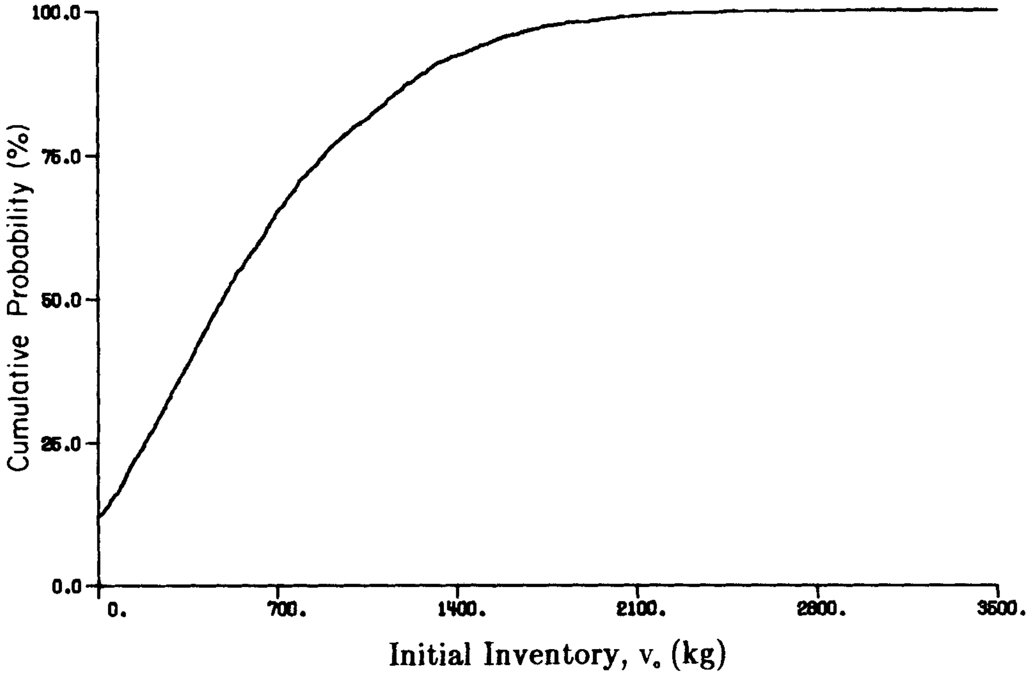

Figure 3. Distribution function for V_0 computed from simulation.

the simulation. As expected, the expected continuity of operation decreases with an increase in T. Note that the design with $V^* =$ 11,746 kg will have expected continuity of at least 90% up to nearly three times the specified 125 h duration of operation. This is another measure of the conservative nature of the results of the analysis of this paper. The overdesign in the result from our analysis is due the assumption that the worst variations are occurring simultaneously. In simulation, however, the variations arise randomly and thus may cancel the effects of each other. The Monte Carlo simulation for this example was done by a computer code of about 500 lines in Fortran 77 using discrete simulation techniques. The 2,000 simulation runs of 125 h duration required CPU time of about 170 s on a VAX 11/780 computer. As one would expect, the time required for simulation increases linearly with the length of simulation. Of course, one also needs to generate the relevant marginal distribution functions to estimate the storage size with a given confidence on continuity. In light of these facts our result, due to its simplicity, clearly offers a better approach than simulation, especially for incorporation of the effects of variability at the preliminary design level.

DISCUSSION

In this section we briefly consider the extension of the results of this paper to a special class of L-M systems and the application of the sizing expressions to multitank or multiproduct operations. We conclude with remarks on the possibilities of active strategies for accommodating process variations.

L-M Systems with Symmetric Delays

The results of this paper are primarily developed for serial systems. However, in the literature a special type of L-M has received attention because of its analytical similarities to a serial system. This is the L-M system with identical parallel units in a stage, with the starting moments of upstream stages separated by time intervals of ω_1/L, and with the starting moments of downstream stages separated by ω_N/M. This choice of starting moment distribution has been termed a symmetric delay time schedule (Karimi and Reklaitis, 1983). Takamatsu et al. (1984) considered process variability in such a system by assuming that the unit schedules can be deliberately readjusted to retain the symmetric delay time schedule whenever a starting moment revision of a unit occurs. There are certain aspects of this system that require careful scrutiny.

First of all, an L-M system with symmetric delays is not always equivalent to a serial system because the flow rate functions for both systems are in general different. As shown by Karimi and Reklaitis (1985), the stage flow rate functions for an L-M system with symmetric delays are given by,

$$F_j^*(t) = \begin{Bmatrix} c_j^*(\mu_j^* + 1)U_j^* & i\omega_j^* \le t \le (i + x_j^*)\omega_j^* \\ c_j^*\mu_j^*U_j^* & (i + x_j^*)\omega_j^* < t < (i + 1)\omega_j^* \end{Bmatrix}$$

where, $j = 1, 2$ $U_1^* = U_1$, $U_2^* = U_N$, $c_1^* = 1$, $c_2^* = -1$, $\omega_1^* = \omega_1/L$, $\omega_2^* = \omega_N/M$, $\mu_1^* = \mathrm{trunc}(Lx_1)$, $\mu_2^* = \mathrm{trunc}(Mx_N)$, $x_1^* = \mathrm{mod}(Lx_1,1)$, and $x_2^* = \mathrm{mod}(Mx_N,1)$. Clearly these flow rate functions are equivalent to those of a serial system only if $\mu_1^* = \mu_2^* = 0$, that is, $x_1 \le 1/L$ and $x_N \le 1/M$. These constraints must be imposed whenever results derived for the serial system are applied to the identical unit L-M system with symmetric delays.

Secondly, there are additional impediments to applying serial system results to analyze advances of starting moments in L-M systems with symmetric delays. Even if the units in a stage are identical, an advance in the starting moment of one unit does not in general guarantee the feasibility of a similar advance for the remaining units in that stage. Thus, in general it is difficult to insure maintenance of a symmetric delay time schedule under general variations of unit starting moments which include advances. In the case of variations which only involve delays of unit starting moments, readjustment of unit schedules to maintain symmetry of operation is in general possible. This case may, however, be too restrictive to be of practical significance.

Multiple Storage Vessels and Multiproduct Systems

The developments in this paper assumed that there was a single storage vessel between stages and that the successive batches of material were homogeneous and thus could be mixed in storage. In principle, there is nothing in the analysis that precludes subdivision of the total intermediate storage capacity, calculated using the sizing expression, among separate physical vessels. The only assumption implicit in the use of our results under such a subdivision of the total storage capacity is that the time required to switch between individual storage tanks as they are filled or emptied is negligible compared to the total transfer time.

As noted in Karimi and Reklaitis (1984), the design of intermediate storage in multipurpose or multiproduct processes in general requires the joint consideration of product scheduling and storage assignment. However, the results reported here can be used in selected instances of multiproduct production. For instance, if storage is shared by a group of products produced in sequential single-product campaigns, then the required intermediate storage capacity can be chosen as the maximum of the sizes required for each individual product. Similarly, suppose a plant is assigned an intermediate storage vessel for each product and single batches of product are produced sequentially in reoccurring cycles. In this case, each tank can be sized using the single product serial system results by merely redefining the cycle time of each product to include the cycle times of the remaining products as idle time (see Karimi and Reklaitis, 1984, for further details).

Of course multistage, serial systems with multiple intermediate storage locations are analyzed by applying the results of this paper sequentially to each successive pair of processing units that have storage allocated between them.

Inventory Maintenance Strategies

In this work, we addressed the issue of determining the initial inventory level V_0 and storage capacity V^* required to assure continuity of operation under process variations without considering how V_0 should be generated or how variations might be compensated by active process variable readjustments. Clearly, from an operational point of view $V_0 > 0$ implies that either initial inventory must be saved between campaigns or that operating parameters must be adjusted so as to accommodate V_0. The determination of the most appropriate strategy is an important operating issue that is yet unresolved but merits further study.

A related operational issue is that of active control of the inventory level in the storage vessel so as to accommodate variations. Our analysis assumed that all the variations occurred randomly and that there were no deliberate and systematic injections of process parameter variations for strategic purposes. In principle, one can manipulate starting moments, batch sizes, or transfer flow rates to dynamically control the holdup in the tank in such a manner that V^* is reduced. For instance, by deliberate injections of appropriate delays, one could maintain $t_{2k_2} - t_{1k_1}$ within a narrow range, thus decreasing both V_0 and V^* while incurring some loss in the overall production rate. These types of active inventory maintenance strategies are certainly of some practical interest but are beyond the scope of the present work.

NOTATION

a_i = starting moment of ith variation
b_i = ending moment of ith variation
c_i = a coefficient assigned to ith unit by Eq. 3 of Part I
EX = expected value of RV X
f = function defined by Eq. 9a
$F(y)$ = distribution of RV Y
$F_i(t)$ = transfer flow function for ith unit
g = function defined by Eq. 9b
$I(t)$ = integral of sum of flow rates
k = number of variations
k_{ij} = number of jth type of variation for ith unit; $j = 1$ starting moment revision, $j = 2$ flow rate variation, $j = 3$ transfer fraction variation
L = number of units in the upstream stage
m = parameter defined in Table 1
M = number of units in the downstream stage
N = total number of units, $L + M$
N_i = number of cycles of ith unit
p,r,R,s = parameters defined in Table 1
$S_{\max},S_{\min}$ = parameters defined in Table 1
S_n = sum of n RV's
t = time
t_{i0} = initial starting moment of ith unit
t_{ij} = t_{i0} + sum of j starting moment revisions of ith unit
T = time units of uninterrupted operation
T_e = time required to empty a batch unit
T_f = time required to fill a batch unit
T_i = shutdown time for a semicontinuous unit
T_p = preparation time and waiting time for a batch unit
T_B = processing time of a batch unit
T_S = processing time for a semicontinuous unit
U_i = nominal transfer flow rate of ith unit
U_i' = modified transfer flow rate for ith unit in a cycle
$V(t)$ = holdup in the storage vessel
V_0 = initial inventory in storage tank
V_i = nominal batch size of ith unit
$V_{\max}$ = maximum value of $I(t)$ defined by Eq. 4
$V_{\min}$ = minimum value of $I(t)$ defined by Eq. 4
V^* = capacity of storage vessel
$V^i(t)$ = holdup function after completion of ith variation without any further variation
x = parameter defined in Table 1
x_i = nominal transfer fraction for ith unit
x_i' = modified transfer fraction for ith unit in a cycle
X,X_i = random variables
y_2,y_2' = parameters defined in Table 1
z_2,z_2 = parameters defined in Table 1
Z = standard normal random variable

Greek Letters

β_i = integers such that $\beta_1\omega_1 = \beta_2\omega_2$
Δt = cumulative sum of starting moment revisions
Δt_i = cumulative sum of starting moment revisions of ith unit
Δt_i^j = amount of jth starting moment revision of ith unit
ΔU_i = cumulative sum of flow rate variations of ith unit
ΔU_i^j = amount of jth flow rate variation of ith unit
ΔV = cumulative sum of batch size variations
ΔV_0 = increment in initial inventory
ΔV_i = cumulative sum of batch size variations of ith unit
ΔV_i^j = amount of jth batch size variation of ith unit
ΔV^* = increment in storage size
Δx_i = cumulative sum of transfer fraction variations of ith unit
Δx_i^j = amount of jth transfer fraction variation of ith unit
$\Phi(\cdot)$ = standard normal distribution function
$\sigma(X)$ = standard deviation of RV X
γ = fraction of a nominal value
ω_i = nominal cycle time of ith unit
ρ = likelihood of an event
τ,θ,ζ = dummy variables or coefficients
χ_1 = partial cumulative sum of starting moment revisions, an RV

Subscripts

L = lower bound
U = upper bound

Mathematical Symbols

max[ ] = maximum of the quantities within the brackets
min[ ] = minimun of the quantities within the brackets
mod(x,y) = z such that $x = ky + z$ with k integer and $0 \le z < y$
$Pr\{x\}$ = probability of event x
sign(x) = $x/|x|$
trunc(x,y) = the greatest integer multiples of y in x
$| \ |$ = absolute value

APPENDIX I: DERIVATION OF EQ. 7

Let us first express $F_2(t)$ in terms of a real (sine-cosine) Fourier series, as follows,

$$F_2(t) = -\frac{V_2}{\omega_2}\sum_{n=1}^{\infty}\frac{2U_2}{n\pi}\sin n\pi x_2 \cos 2n\chi\left(\frac{t}{\omega_2}-\frac{x_2}{2}\right)$$

Let $I_2(t)$ denote the integral of $F_2(t)$ as,

$$I_2(t) = \int_0^{-t} F_2(\tau)d\tau$$

Integrating $F_2(\tau)$ term by term,

$$I_2(t) = \frac{V_2}{\omega_2}t - \sum_{n=1}^{\infty}\frac{U_2\omega_2}{n^2\pi^2}\sin n\pi x_2\left[\sin n\pi x_2 - \sin 2n\pi\left(\frac{t}{\omega_2}+\frac{x_2}{2}\right)\right]$$

From (Tuma, 1979),

$$\sum_{n=1}^{\infty}\frac{\cos 2n\pi\theta}{n^2\pi^2} = \frac{1}{6} - |\theta| + \theta^2 \, |\theta| \le 1 \qquad \text{(A1)}$$

Define $y_2 = \mathrm{mod}(t,\omega_2)/\omega_2$. Expressing $I_2(t)$ in terms of cosine functions and using Eq. A1, we obtain,

$$I_2(t) = \frac{V_2}{\omega_2} - \frac{V_2}{2}(1-x_2) - \sum_{n=1}^{\infty}\frac{U_2\omega_2}{2n^2\pi^2} \times \cos 2n\pi(x_2+y_2-1) - \cos 2n\pi y_2]$$

Since, $0 \le y_2 \le 1$ and $|x_2 + y_2 - 1| \le 1$, using Eq. A1, $I_2(t)$ can be rewritten as

$$I_2(t) = V_2\left[\frac{t}{\omega_2} - y_2\right] + \frac{U_2\omega_2}{2}[x_2 + y_2 - 1 + |x_2 + y_2 - 1|]$$

Defining $s = \max(0, x_2 + y_2 - 1)$ and taking $t = t_{2k_2} - t_{1k_1}$, Eq. 7 results.

APPENDIX II: PROOF OF LEMMA 1

Let us define $\tau = (t_{2k_2} - t_{1k_1})/\omega_2$. It will suffice to prove the lemma for $f(\tau)$ since the proof for $g(\tau)$ is very similar. Also define,

$$f_1(\tau) = \tau \text{ and } f_2(\tau) = \frac{RS_{\max}}{x_2} - y_2'$$

Recall that $y_2 = \text{mod}(\tau,1)$ and $y_2' = \text{mod}(y_1 - \delta_1,p)$, hence $f_2(\tau)$ is periodic with period p. Clearly, $f_2(\tau) \equiv f_2(y_2')$ and hence, for $0 \le y_2' \le p$, we have,

$$f_2(y_2') = \begin{cases} -y_2' & 0 \le y_2' \le p(1-x) \\ \dfrac{R[y_2' - p(1-x)]}{x_2} - y_2' & p(1-x) \le y_2' < p \end{cases}$$

Therefore, $f_2(0) = 0$ and $f_2(y_2')$ is piecewise linear with slope of -1 for $0 \le y_2' \le p(1-x)$ and positive slope for $p(1-x) \le y_2' < p$. Since, $f_1(\tau)$ has a slope of $+1$, it is clear that $f(\tau)$ always has a nonnegative slope.

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